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市場與信用風險交互影響下風險性證券之易脆選擇權的評價避險

Pricing and Hedging Strategies of Vulnerable Black-Scholes Option on Defaultable Securities Subject to the Intersection of Market and Credit Risk

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摘要

本文旨在探究當存在市場與信用風險之交互影響下風險性證券之易脆選擇權的評價與避險。本文模型假設違約強度函數服從二因子 Cox 過程，並同時考量交易對手與標的資產違約風險間的交互性。本研究呈現之結果如下：首先，交易對手違約風險導致了權證價格上的信用折耗；然而，標的資產違約風險對選擇權價值產生信用補貼。標的資產違約風險效果明顯占優於對手違約風險效果。其次，此兩不同種類違約風險對權證避險比率亦造成截然不同之影響。究其原因，標的資產違約風險對權證所造成之不對稱影響係因正向「槓桿效果」與負向「提前保值效果」之間的抵換均衡於買權與賣權此二情況下並非一致所導致。另外，給定不同水準下的總體經濟變數，本文模型證實標的物違約風險效果與對手違約風險效果之間存在正向的共移性。此發現可對應實證文獻所探討的「信用感染」及「成串違約」等現象之經濟意義。

關鍵詞：混合選擇權、標的資產違約風險、交易對手違約風險、自發性違約強度

Abstract

This paper derives the pricing formulas and put-call parity for "*hybrid options*"—vulnerable options on defaultable securities—with the presence of the intersection of market and twofold default risk. Default intensities are modeled as a two-factor Cox process, and the dependency of option writers' default on the underlying stock default is also captured. We find counterparty risk generates credit discount on option value, while reference risk generates credit premium. The latter significantly dominates the former. The impact of twofold default risk on option hedging ratios fails to replicate consistency in the price pattern. The

asymmetric behavior of option deltas with respect to reference risk is attributed to the trade-off between negative early-fixed effect and positive leverage effect. Also, our model generates a positive co-movement between these two types of default risk impact with varied choices of economic variables. Such a model feature reflects the empirical implication of credit contagion as well as the clustering of default.

Keywords: Hybrid Options, Reference Risk, Counterparty Risk, spontaneous default intensity

1.INTRODUCTION

Since the well-known Black-Scholes formula proposed in 1973, option pricing has received considerable attention from academic research.[1] As credit events emerge in an endless stream recently, the intersection of default risk and financial markets has gradually played an important role in financial engineering and risk management.[2] Generally, there are two sources of credit risk. The first is "counterparty credit risk" introduced by the issuer of over-the-counter (OTC) derivatives, who may default due to insufficient margin in such transactions. Another, termed "reference risk", is the one where risky asset underlying the derivative may default; paying off less than promised, and thus such a risk is that part of contractual risk involved with third party. However, traditional theories

[1] Prior works explore option pricing in several dimensions; such as stochastic term structure of interest rate, asset illiquidity, volatility structure, market incompleteness, and path-dependent features. For example, see Scott (1987); Kunitomo & Ikeda (1992); Heston (1993); Hull & White (1995); Geman & Yor (1996); Rich (1996); Barone et al. (1998); Baldi et al. (1999); Klein & Inglis (1999); Pelsser (2000); Liao & Wang (2002); Guillaume (2003); Hui et al. (2003); Hung & Liu (2005); Cetin et al. (2006); and Lo & Hui (2007).

[2] In recent years, prominent credit events include Global financial crises (through 2007 and 2008), Sub- prime mortgage crises (late 2006), and European credit crises (from 2011). Several earlier examples of credit events contain East-Asia banking crisis (in 1997) and Long-term Capital Management's potential defaults (in 1998).

largely work with an unrealistic assumption that option trading is always default-free. Motivated by such irrationality, a class of papers thus devotes to understanding the impact of default risk on option price.[3]

Johnson & Stulz (1987) pioneers in option pricing subject to default risk with using structural model of Merton (1974). They name a standard option with unilateral counterparty risk as a vulnerable option-namely, the option contract in which its writer may possibly default on his obligations, and thus a vulnerable option whose payoff at expiration depends on whether the default event of option writer occurs or not during the contractual lifespan. That article also shows that the value of vulnerable call is not only always less than the corresponding standard call, but falls with time to maturity, the spot interest rate, and the volatility of underlying asset. Following Johnson and Stulz, subsequent works putting efforts in this subject include Hull & White (1995); Klein (1996); Rich (1996); Klein & Inglis (1999, 2001); and Hui et al. (2003).

These studies consistently suffer two aspects of drawbacks. First, they are solely concerned with default likelihood of option writer. Such doing, however, substantially underestimates the effect of credit risk. In reality, there exist many financial securities embedded with options that would be credit-risky or defaultable (e.g., corporate non- callable bond or common stock). Roles played by reference risk in option pricing thus should not be ignored, and may even dominate over counterparty risk.

Second, the structural approach of modeling default itself, widely used by these studies, has restrictions on the application. For instance, some of relevant parameters (e.g., firm's total asset value) are often non-observed and not easily estimated. This naturally raises difficulties for analyses. Besides, the intersection of macroeconomic factors and default risk fails to be captured.[4] Since within structural approach defaults are treated as an endogenous process, combining firm-value dynamics with the state of macro-economy might face significant modeling

[3] For notational consistency, throughout this study we term reference/counterparty risks as "twofold default risk" and an option featured by this type of risk as a "hybrid option".

[4] For empirical evidence that the changes in debts' credit spreads can be explained by macroeconomic factors, please refer to Altman (1983); Wilson (1997); Kao (2000); Campbell & Taksler (2003); and Huang & Kong (2003).

difficulties.

To simultaneously improve these deficiencies and explore our central issue, we adopt the intensity-based approach proposed by Jarrow & Turnbull (1995) in deriving the value of vulnerable claims. Our model has three aspects of attractive features:

1. Default intensity follows a two-factor Cox process (two proxy variables: default-free spot interest rate and market index return) that well captures the randomness of default process;[5] Such an idea not only makes the extension of Lando (1998) and Jarrow & Turnbull (2000), but also helps us combine option pricing with the intersection of macro economic factors and twofold default risk.
2. The dependence between reference and counterparty risks is considered; We study a case where the option writer's spontaneous default intensity depends on whether the underlying security defaults before the option's maturity. In this way our model allows for examining the co-movement between twofold default risks, and also can echo with empirical implications of credit contagion and the clustering of default (Kraft & Steffensen, 2007; Jorion & Zhang, 2009).
3. The availability of the proposed model is multiple; It allows for pricing the hybrid option, vulnerable option, non-vulnerable option with reference risk, and default- free vanilla option in a unified framework. To demonstrate the comparison among options with different types of credit risk, Table 1 is given as below.

[5] Lando (1998) illustrates how a doubly stochastic Poisson process progresses a framework for pricing financial securities subject to dependency between credit and market risks. Jarrow & Turnbull (2000) integrate market risk into credit risk by setting up the intensity function as a two-factor Cox process to represent the dependence of the probability of default on the state of economy. They pick two proxy variables: the spot default-free interest rate and the uncertain fluctuation of market index return for mentioned intensity functions, because moves in these two variables are easily visual on a daily basis, unlike other economic variables reported quarterly or semiannually.

Table 1 Model-Implication Comparison

	With Reference Risk	Without Reference Risk
With Counterparty Risk	Hybrid option [Our paper]	Vulnerable option [Klein & Inglis (1999)]
Without Counterparty Risk	Non-vulnerable option on defaultable securities [Our paper]	Standard option [Heath et al. (1992)]

Data source: Heath et al. (1992), Klein & Inglis (1999), and this paper.

Main findings of this study add to option-pricing literature in three ways. First, the impact of counterparty and reference risk on option price are entirely different. The former generates credit discount, consistent with most of existing literature (Johnson & Stulz, 1987), but the latter surprisingly generates credit premium reflecting the tradeoff between leverage effect and early-fixed effect. In our model reference risk impacts option value in two ways. On the one hand, reference risk accompanied with debt financing makes the underlying asset's return (common stock or equity) more volatile via financial leverage, and hence, results in a positive impact on option price (termed as leverage effect). On the other hand, reference risk delivers possibilities that the expected intrinsic value of option could be fixed at times earlier than its expiration, leading to a negative/positive impact on call/put price (termed as early-fixed effect).

Second, the behavior exhibited by option deltas with respect to twofold default risk makes a coherent response to option price. Specifically, counterparty risk reduces deltas to discourage option issuers' hedging intention; while reference risk increases (decreases) deltas to encourage (discourage) hedging intention when leverage (early- fixed) effect dominates (leverage) early-fixed effect. The inverse interaction between deltas and counterparty risk means that, instead of concerning about whether perfect hedge can be reached, option issuers being more likely to default will put less effort in hedging activities. The implication behind asymmetric impacts of reference risk on deltas is: leverage (early-fixed) effect raises (lowers)

the variance of expected intrinsic value of option such that the interference of this effect in option trade enlarges (saves) issuers' expenses on hedging against the fluctuation in underlying stock price.

Third, there exists positive co-movement between reference risk and counterparty risk on the option price. Also, option price behaves more sensitive to reference risk than counterparty risk. Given the explicit separation between the two types of risk effects on price, we show that percentage changes in price due to counterparty risk are much lower than due to reference risk. Besides, the net effects of reference risk and counterparty risk both are stronger if choosing a lower level for economic variables, such as market index. Such a model feature agrees with empirical implication of credit contagion as well as the clustering of default.

The rest of this study is structured as follows. In Section 2 below, we construct a general model for pricing options in the presence of the mixture of twofold default risk and market randomness. Section 3 numerically examines the characters of various types of vulnerable options. Conclusion is drawn in Section 4. Technical details are relegated to the appendix.

2.MAIN MODEL

Throughout the paper we fix a filtered probability space (Ω, G, P), and also consider a continuous-trading economy endowed with filtration $\mathrm{G} = (G_t)_{t\geq 0}$ that satisfies $\mathrm{G} = \mathrm{F}^{\mathrm{W}} \vee \mathrm{H}$ (i.e., $G_t = F_t^W \vee H_t$ for any $t \in \Re_+$) and is rich enough to support whole stochastic processes under the real measure P.

2.1 Preliminaries: Default, Price Dynamics, and Information Structure

By following Jarrow & Turnbull (1995) and Lando (1998), this research defines default as a strictly-positive random time $\tau_j : \Omega \to \Re_+$: with convention $\inf \phi = +\infty$ and for $j = i \; or \; S$

$$\tau_j := \inf(t : \int_0^t \lambda_j^{\mathrm{P}}(u)du \geq E_j); \;\; \Lambda_j^{\mathrm{P}}(t) = \int_0^t \lambda_j^{\mathrm{P}}[r(u), \ln(I(u)/B(u))]du; \;\; \forall t \geq 0 .$$

Specifically, τ_S and τ_i denotes the random time of default on credit-risky under- lying asset price $S(\cdot)$ and option issuer respectively. Both the two random time can be interpreted as a first jump time of Cox process $N^j(\cdot)$ with non-negative hazard functions $\Lambda_j^{\mathrm{P}}(\cdot)$. E_j is a unit exponential random variable. Term in the integral $\lambda_j^{\mathrm{P}}[\cdot]$ represents the strictly positive default intensity with respect to F^W. As the default intensity becomes large, the integrated hazard will rise up faster and touches the level of independent exponential variable earlier; eventually, it leads the random time to be small and urges the probability of default to become higher. This mentioned intensity function is explicitly formed as a linear combination of spontaneous default intensity $(\hat{\alpha}_0, \hat{\lambda}_0)$, the current level of the spot interest rate $r(\cdot)$ and the abnormal market index return $\ln(I(\cdot)/B(\cdot))$[6]

$$\begin{cases} \lambda_i^{\mathrm{P}}[r(u), \ln(I(u)/B(u))] = \lambda_i^{\mathrm{P}}(u) = \hat{\alpha}_0 + \hat{\alpha}_1 r(u) + \hat{\alpha}_2 \ln(I(u)/B(u)) & (1) \\ \lambda_S^{\mathrm{P}}[r(u), \ln(I(u)/B(u))] = \lambda_S^{\mathrm{P}}(u) = \hat{\lambda}_0 + \hat{\lambda}_1 r(u) + \hat{\lambda}_2 \ln(I(u)/B(u)) & (2) \end{cases}$$

In expressions (1) and (2), all coefficients $\hat{\alpha}_1, \hat{\lambda}_1$ and $\hat{\alpha}_2, \hat{\lambda}_2$ are fixed and measure the sensitivity of intensity function with respect to spot rate and cumulative abnormal return on the market index respectively.

The implication behind expressions (1) and (2) is supported by much of existing literature on credit risk, which argue that macroeconomic factors do have significant power in predicting default likelihood and credit spread on corporate bonds. Duffee (1998) finds a reverse correlation between three-month Treasury yields and corporate- bond yield spreads. Kao (2000) concludes that Russell 2000 index return and interest rate both display significant interpretive power for features of credit spreads. Campbell & Taksler (2003) find that the equity volatility can explain about one-third of the variation in yield spreads. Huang & Kong (2003)

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[6] For tractability, consider time-0 market index as $I(0) = 1$, and $B(t)$ denotes the wealth accumulated by an initial one-dollar investment at spot rates in each subsequent period. Because the initial saving account $B(0) = 1$, the term $\ln(I(u)/B(u)) = \ln(I(u)/I(0)) - \ln(B(u)/B(0))$ measures the geometric abnormal return on market index.

find that high interest rates and steep yield curves are often associated with an expanding economy, and low credit spreads as well as the higher interest rate volatilities are usually associated with wider credit spread, especially for high-yield bond indexes.

To clarify the information (filtration) structure, now introduce a right-continuous first jump process $\mathcal{H}^j(t) = 1_{(\tau_j \le t)} = N^j(t \wedge \tau_j)$. Such a process will stop at τ_j and generates $\sigma-$algebra $H_t^j = \sigma(\mathcal{H}^j(u), 0 \le u \le t)$ with filtration $(H_t^j)_{t\ge 0} = \mathrm{H}^j$. Therefore, sub-filtration $\mathrm{H} = \mathrm{H}^S \vee \mathrm{H}^i$ collects information on the observation of default up to current time, and the compensated Poisson process $\hat{N}^j(u) = N^j(t \wedge \tau_j) - \Lambda_j^{\mathrm{P}}(t \wedge \tau_j)$ defined on (Ω, G, P) also follows a $\mathrm{G}-$martingale.

The other sub-filtration $\mathrm{F}^W = \mathrm{F}^S \vee \mathrm{F}^r \vee \mathrm{F}^I \vee \mathrm{F}^B$ simultaneously contains the information about the observation on the market value of defaultable underlying stock $F_t^S = \sigma(S(u), 0 \le u \le t)$, on the riskless spot interest rate $F_t^r = \sigma(r(u), 0 \le u \le t)$, on the market index $F_t^I = \sigma(I(u), 0 \le u \le t)$, and on the saving account $F_t^B = \sigma(B(u), 0 \le u \le t)$. The associated dynamics of $S(\cdot)$, $r(\cdot)$, $I(\cdot)$, and $B(\cdot)$ are displayed as:[7,8]

$$dS(t)/S(t-) = (\mu^S - q)dt + \hat{\sigma}_s \cdot dW^{\mathrm{P}}(t) - d[N^S(t \wedge \tau_s) - \Lambda_s^{\mathrm{P}}(t \wedge \tau_s)], \quad (3)$$

[7] Jarrow & Turnbull (1995) extend the idea of Merton (1976) to define the stochastic value dynamics for defaultable common equity as a linear combination of a compensated Poisson process and geometric Brownian motion. In addition to common equity, non-convertible corporate bond is another type of securities suffering reference risk. Notably, because a finitely-matured bond is non-tradable after its expiration, expression (3) can only be applied to a perpetual risky bond without explicit maturity date. See Leland (1994), Chen (2010), and Chen et al. (2010) for a detailed description about the features of a consol debt and the explicit form of its value dynamics.

[8] In expression (3) the reference-type default on the underlying stock is modeled as a one-time jump process. In this way the price of underlying stock suddenly falls to zero once the reference-type default occurs, which follows the feature of absolute priority that, after liquidation, there leaves no remaining value for equity holders. However, such doing does not make the sense for options on a corporate bond with non-zero recovery payment. An effective way to overcome this problem is to impose a terminal condition on the value dynamics of underlying assets. Our assumption of one-time jump in fact plays a special case of multi-jumps model of Merton (1976). But the model implications are different. Merton interprets multi-jumps on the stock-price dynamics as the price discontinuities caused by market-based abnormal shock, while in this study the one-time jump is used to capture how liquidation accompanied with individual default freezes the trading with corresponded equity.

$$dr(t) = a[\vartheta(t) - r(t)]\,dt + \hat{\sigma}_r \cdot dW^{\mathrm{P}}(t), \tag{4}$$

$$dI(t)/I(t) = \mu^I dt + \hat{\sigma}_I \cdot dW^{\mathrm{P}}(t), \tag{5}$$

$$dB(t)/B(t) = r(t)\,dt$$

$$\hat{\sigma}_s = \sigma_s \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \hat{\sigma}_r = \sigma_r \cdot \begin{bmatrix} \rho_{Sr} \\ \sqrt{1-\rho_{Sr}^2} \\ 0 \end{bmatrix}, \hat{\sigma}_I = \sigma_I \cdot \begin{bmatrix} \rho_{IS} \\ (\rho_{Ir} - \rho_{rS} \times \rho_{IS}) \big/ \sqrt{1-\rho_{rS}^2} \\ \sqrt{1-\rho_{IS}^2 - (\rho_{rI} - \rho_{rS} \times \rho_{IS})^2 \big/ 1-\rho_{rS}^2} \end{bmatrix}, W^{\mathrm{P}}(t) = \begin{bmatrix} W_S^{\mathrm{P}}(t) \\ W_r^{\mathrm{P}}(t) \\ W_I^{\mathrm{P}}(t) \end{bmatrix}$$

where μ^S and μ^I indicate the constant appreciation rate of the underlying stock price and the market index respectively; q is dividend paying ratio; $\vartheta(\cdot)$ denotes a deterministic function for fitting initial term structure with constant parameter a (see Heath et al., 1992); σ_s, σ_r, and σ_I respectively denotes the instantaneous volatility for $S(\cdot), r(\cdot)$, and $I(\cdot)$ over the trading period $[t, T]$; $W^{\mathrm{P}}(\cdot)$ is a three-dimensional standard Wiener process defined on the filtered probability space $(\Omega, F^W, \mathrm{P})$; ρ_{Ir}, ρ_{Is}, and ρ_{rs} symbolizes the correlation between W_r^{P} and W_I^{P}, W_I^{P} and W_S^{P}, and W_r^{P} and W_S^{P} respectively.

2.2 Measure Change: Incorporating the Interaction between Counterparty Default and Reference Default into Stock Price Dynamics

We model default as an exogenous one-time jump process related to macro-economic state variables, including interest rates and market index return. In this way the positive interaction between reference-type and counterparty default does exist within economy. Such a concept echoes with empirical evidences of credit contagion as well as the clustering of default (Kraft & Steffensen, 2007; Jorion & Zhang, 2009).

To capture the contagion effects of reference default on counterparty default in pricing the option, we follow the counter-example of Kusuoka (1999). We consider a particular martingale measure P^* equivalent to P on space (Ω, G) that allows for examining the random times mentioned above. Under this measure the

counterparty's spontaneous intensity of random default time τ_i moves (or switches) from the value $\lambda_i[r(u),\ln(I(u)/B(u))]$ to another specific value $\zeta_i[r(u),\ln(I(u)/B(u))]$ as soon as τ_S occurs. The measure P^* can be introduced by defining the corresponded Radon- Nikodym density process $\hbar(u), t \le u \le T$ as:

$$\hbar(u) := \left.\frac{d\,\mathrm{P}^*}{d\,\mathrm{P}}\right|_{G_u}, \text{ P - a.s.}$$

with integral representation

$$\hbar(T) = \hbar(0) + \int_{[0,T]} \hbar(u-)[\gamma_u\, dW^{\mathrm{P}}(u) + \kappa_u^i\, d\hat{N}^i(u) + \kappa_u^S\, d\hat{N}^S(u)].$$

Thus, via Girsanov's theorem, we have

$$W^*(t) = W^{\mathrm{P}}(t) - \int_0^t \gamma_u\, du\,;\quad \Lambda_j^*(t) = \int_0^t (1+\kappa_u^j)\, d\Lambda_j^{\mathrm{P}}(u)\text{; and}$$

$$\lambda_j^*(u) = (1+\kappa_u^j)\lambda_j^{\mathrm{P}}(u).$$

For fitting P^* as a risk-neutral measure, F^W-predictable process γ_u must be solved by the following system of equations

$$\begin{cases} r(u) - \mu^S - \gamma_u \cdot \hat{\sigma}_S = 0 \\ a[\iota(u) - \vartheta(u)] - \gamma_u \cdot \hat{\sigma}_r = 0 \\ r(u) - \mu^I - \gamma_u \cdot \hat{\sigma}_I = 0 \end{cases}$$

and $\mathrm{F}^W \vee \mathrm{H}^S$-predictable κ_u^i and F^W-predictable κ_u^S respectively has a unique form

$$\kappa_u^i = \frac{1_{\{\tau_s<u\}}\zeta_i[r(u),\ln(I(u)/B(u))] + 1_{\{\tau_s\ge u\}}\lambda_i[r(u),\ln(I(u)/B(u))] - \lambda_i^{\mathrm{P}}[r(u),\ln(I(u)/B(u))]}{\lambda_i^{\mathrm{P}}[r(u),\ln(I(u)/B(u))]}$$

$$\kappa_u^S = \frac{\lambda_S[r(u),\ln(I(u)/B(u))] - \lambda_S^{\mathrm{P}}[r(u),\ln(I(u)/B(u))]}{\lambda_S^{\mathrm{P}}[r(u),\ln(I(u)/B(u))]};$$

where

$$\zeta_i\,[\,r(u), \ln(I(u)/B(u))\,] = \zeta_0 + \zeta_1 r(u) + \zeta_2 \ln(I(u)/B(u))\,;$$

$$\lambda_i\,[\,r(u), \ln(I(u)/B(u))\,] = \alpha_0 + \alpha_1 r(u) + \alpha_2 \ln(I(u)/B(u))\,;$$

$$\lambda_S[\,r(u), \ln(I(u)/B(u))\,] = \lambda_0 + \lambda_1 r(u) + \lambda_2 \ln(I(u)/B(u))\,.$$

Applying Girsanov's theorem and solutions for κ_u^i, κ_u^S, and γ_u to expressions (3)-(5) can yield the risk-neutral dynamics of stock price, spot interest rate, and market index return as below:

$$dS(t)/S(t-) = [\,r(t) - q\,]\,dt + \hat{\sigma}_s \cdot dW^*(t) - d[\,N^S(\,t \wedge \tau_s) - \Lambda_s^*(\,t \wedge \tau_s)\,]\,;$$

$$dr(t) = a[\,\iota(t) - r(t)\,]\,dt + \hat{\sigma}_r \cdot dW^*(t)\,;$$
$$dI(t)/I(t) = r(t)\,dt + \hat{\sigma}_I \cdot dW^*(t)\,.$$

In the next section we repeatedly use the SDEs above in deriving the option pricing formulas.

2.3 Pricing the Hybrid Call

Suppose that markets are perfect and frictionless, there are no taxes, transaction costs, and dividend payments ($q = 0$), and all financial derivatives are tradable continuously.[9] Consider a circumstance of option issuance where a representative risk neutral agent has limited paying capacity and intends to sell European calls on a credit -risky common stock formed as equation (3). The contractual payoff of this hybrid call with predetermined maturity date T, strike price K, and recovery rate $\delta \in [0,1]$ is structured as below

$$C(T) \equiv \begin{cases} S(T) - K & \text{if } \tau_S > T, \tau_i > T, S(T) > K \\ \delta[\,S(T) - K\,] & \text{if } \tau_S > T, T \geq \tau_i, S(T) > K \\ 0 & \text{otherwise} \end{cases}$$

or

[9] Because this study is aimed at examining the impacts on option price of the intersection of twofold default risk and macroeconomic factors, we do not attempt to address the issues on dividend payments, and thus, simplify our analysis with letting $q = 0$.

$$C(T) \equiv [S(T)-K]^{+} 1_{\{\tau_s > T\} \cap \{\tau_i > T\}} + \delta [S(T)-K]^{+} 1_{\{\tau_s > T\} \cap \{t < \tau_i \le T\}} \qquad (6).$$

The contingent payoff (6) has its economical implication; the option-holder receives the whole intrinsic value of call $[S(T)-K]^{+}$ only if the writer maintains his sufficient solvency ($\tau_i > T$) and if the tradability of underlying stock can be hold ($\tau_S > T$) over the whole trading period-otherwise only a portion of intrinsic value can be redeemed $\delta[S(T)-K]^{+}$ once option writer is insolvent at expiration date. Notably, the terminal moneyness of call is out-the-money after the occurrence of credit events associated with underlying stock ($t < \tau_S \le T$), because the underlying stock price at maturity date conditional on reference-type default falls to zero.

To price a call under the intersection of market and credit risk, introducing the default-free zero-coupon bond with paying one dollar at option's maturity date into pricing framework is needed. According to the forward martingale theory of Musiela & Rutkowski (1997), the evolution of this bond price is defined as the following

$$dB(t,T)/B(t,T) = r(t)\,dt + b(t,T) \cdot dW^{*}(t)$$

where $b(t,T) = \hat{\sigma}_r \{\exp[a(t-T)]-1\}/a$.[10] Thus using the forward martingale pricing property yields the expression of current value of the hybrid call under the forward measure $\mathrm{P_T}$:

$$C(t) = B(t,T)\,\mathrm{E}_{\mathrm{P_T}}[\,C(T)/B(T,T) \,|\, G_t\,] \qquad (7)$$

Equation (7) plays the central pricing equation that has explicit form displayed as below.[11]

Proposition 1. Given non-default initial conditions $\tau_S > t$ and $\tau_i > t$, the terminal payoff of a vulnerable call on s credit-risky common

[10] For more details about the forward martingale method and the technique of changing from the spot martingale measure to the forward measure, see Chapter 7 in Musiela & Rutkowski (1997).

[11] See Appendix for a detailed proof.

stock as (6), and the dynamics of underlying stock as expression (3), the current price of a European hybrid call with twofold default risk is as follow.

$$C(t)=(1-\delta)\{S(t)\exp[\int_t^T Z(u)du+(\lambda_1-\alpha_1)U]\mathrm{N}(d_1)-KB(t,T)\exp[-\int_t^T Y(u)du]\mathrm{N}(d_2)\}$$
$$+\delta\{S(t)\exp[\int_t^T X(u)du+\lambda_1 U]\mathrm{N}(d_3)-KB(t,T)\mathrm{N}(d_4)\}$$

where

$$Z(u)=\Theta(u;\lambda_0-\alpha_0,\lambda_1-\alpha_1,\lambda_2-\alpha_2,\eta(u),1,1,0);$$

$$Y(u)=-\Theta(u;-\alpha_0,-\alpha_1,-\alpha_2,\Phi(u),0,0,1);$$

$$X(u)=\Theta(u;\lambda_0,\lambda_1,\lambda_2,\xi(u),1,1,0);$$

$$\eta(u)=\Delta(u;\lambda_1-\alpha_1,\lambda_2-\alpha_2);\quad \Phi(u)=\Delta(u;-\alpha_1,-\alpha_2);\quad \xi(u)=\Delta(u;\lambda_1,\lambda_2);$$

the arguments to standard normal cumulative distribution $\mathrm{N}(\cdot)$ are

$$d_1=\Psi(\beta(u),\lambda_1,\lambda_1-\alpha_1,\lambda_1,\lambda_2);\quad d_2=\Psi(\Gamma(u),0,0,\lambda_1,\lambda_2);$$

$$d_3=\Psi(\theta(u),\lambda_1,\lambda_1,\lambda_1,\lambda_2);\quad d_4=\Psi(\phi(u),0,0,\lambda_1,\lambda_2);$$

$$\beta(u)=\Xi(u;\eta(u),1,0,0,1,\lambda_2-\alpha_2,\hat{\sigma}_S,\lambda_0,\lambda_1,\lambda_2,\eta(v));$$

$$\Gamma(u)=\Xi(u;\Phi(u),-1,1,\alpha_2,0,0,b(v,T),\lambda_0,\lambda_1,\lambda_2,\Phi(v));$$

$$\theta(u)=\Xi(u;\xi(u),1,0,0,1,\lambda_2,\hat{\sigma}_S,\lambda_0,\lambda_1,\lambda_2,\xi(v));$$

$$\phi(u)=\Xi(u;0,-1,0,0,0,0,b(v,T),\lambda_0,\lambda_1,\lambda_2,0);$$

and the combo functions Θ, Δ, Ψ, and Ξ have their respective explicit form

$$\Theta(u;a,b,c,e,g,h,i):=a+b[f(t,u)-g\cdot\hat{\sigma}_S\cdot b(u,T)]+0.5e^2+c\{h\cdot(\hat{\sigma}_I\cdot\hat{\sigma}_S-0.5\left|\hat{\sigma}_I\right|^2)(u-t)$$
$$\ln[I(t)/B(t)]+i\cdot\int_t^u[\hat{\sigma}_I\cdot b(v,T)-0.5\left|\hat{\sigma}_I\right|^2]\,dv\};$$

$$\Delta(u;j,k):=-j\cdot b(u,T)+k\cdot\hat{\sigma}_I\cdot(T-u);$$

$$\Psi(l,m,n,o,p):=(\ln\tfrac{S(t)}{KB(t,T)}+\int_t^T l\,du+m(U+n\left|R\right|^2V))(\int_t^T\left|\hat{\sigma}_s-(1+o)b(u,T)+p(T-u)\cdot\hat{\sigma}_I\right|^2du)^{-\frac{1}{2}};$$

$$\Xi(u;q,s,v,w,x,y,z,\varpi,\varsigma,\upsilon,\tilde{\lambda})$$
$$:=q[\hat{\sigma}_S-b(u,T)]+s0.5\left|\hat{\sigma}_S-b(u,T)\right|^2+\upsilon\{\ln[I(t)/B(t)]-0.5\left|\hat{\sigma}_I\right|^2(u-t)+\int_t^u(z+\tilde{\lambda})\cdot\hat{\sigma}_I\,dv\}+\varpi$$
$$+\varsigma\{f(t,u)-v\alpha_1\left|b(u,T)\right|^2+w\hat{\sigma}_I\cdot b(u,T)(T-u)-x\hat{\sigma}_S\cdot b(u,T)-yR\cdot\hat{\sigma}_I(T-u)[A(u,T)-1]\};$$

$$U:=(\left|R\right|^2/4a)[4A(t,T)-A^2(t,T)-3+2a(T-t)];$$

$$V:=(T-t)+\{[A(t,T)-1]-0.5[1-2A(t,T)+A^2(t,T)]\}/a;$$

$$R:=\hat{\sigma}_r/a;\quad A(t,T):=\exp[a(t-T)];\quad b(t,T):=R(A(t,T)-1).$$

Proposition 1 is useful for examining how the intersection between market and credit risk affects option valuation and what the difference in the sensitivity of option value to default risk between reference and counterparty default is. Its availability is multiple, rather than a pricing formula exclusively for hybrid calls. It not only offers the clues to further derive hybrid put-call parity, but also contains several interesting special cases of option pricing, such as the standard vanilla options. These extensions will be discussed in the following subsections.

2.4 Pricing the Hybrid Put via Hybrid Put-Call Parity

Next consider the valuation of a hybrid put through hybrid put-call parity. The economic implications behind the contingent payoff of a hybrid put are a bit different from call. Because the occurrence of reference default heavily knocks the underlying stock price down to zero, the call shocked by reference default will

become worthless, whereas the put holder can enjoy a certain rebate amount for option's intrinsic value. Without loss of generality, let such a rebate amount be notated as $\rho \in [0, K]$. This key feature of put's moneyness conditional on reference default helps highlight asymmetry in the payoff structure between call and put. The terminal payoff of a corresponded hybrid put thus can be displayed as below

$$P(T) \equiv \begin{cases} K - S(T) & if\ \tau_S > T, \tau_i > T, S(T) < K \\ \delta[K - S(T)] & if\ \tau_S > T, T \geq \tau_i > t, S(T) < K \\ \rho & if\ T \geq \tau_S > t, \tau_i > T \\ \delta\rho & if\ T \geq \tau_S > t, T \geq \tau_i > t \\ 0 & otherwise \end{cases}$$

or

$$\begin{aligned} P(T) \equiv [K - S(T)]^+ 1_{\{\tau_s > T\} \cap \{\tau_i > T\}} + \delta [K - S(T)]^+ 1_{\{\tau_s > T\} \cap \{t < \tau_i \leq T\}} + \rho 1_{\{\tau_i > T\} \cap \{t < \tau_S \leq T\}} \\ + \delta\rho 1_{\{t < \tau_i \leq T\} \cap \{t < \tau_S \leq T\}}. \end{aligned} \tag{8}$$

Putting (6) and (8) together, we yield the hybrid put-call parity

$$\begin{aligned} P(T) = C(T) + [K - S(T)][1_{\{\tau_S > T\} \cap \{\tau_i > T\}} + \delta \cdot 1_{\{\tau_S > T\} \cap \{t \leq \tau_i < T\}}] + \rho 1_{\{\tau_i > T\} \cap \{t < \tau_S \leq T\}} \\ + \delta\rho 1_{\{t < \tau_i \leq T\} \cap \{t < \tau_S \leq T\}}. \end{aligned} \tag{9}$$

Similar to (7), rewriting (9) with the forward martingale property has

$$\begin{aligned} P(t) = C(t) + B(t,T) \mathrm{E}_{\mathrm{P_T}} \{[K - S(T)][1_{\{\tau_S > T\} \cap \{\tau_i > T\}} + \delta \cdot 1_{\{\tau_S > T\} \cap \{t \leq \tau_i < T\}}] \big| G_t\} \\ + B(t,T) \mathrm{E}_{\mathrm{P_T}} \{\rho\, [1_{\{\tau_i > T\} \cap \{t < \tau_S \leq T\}} + \delta \cdot 1_{\{t < \tau_i \leq T\} \cap \{t < \tau_S \leq T\}}] \big| G_t\}. \end{aligned}$$

Again with using the forward martingale method, the explicit solution for $P(t)$ and hybrid put- call parity can be derived. The formula is displayed as Proposition 2 below.

Proposition 2. Given non-default initial conditions $\tau_S > t$ and $\tau_i > t$, the terminal payoff of a vulnerable put on s credit-risky common stock as (8), and the dynamics of underlying stock as expression (3), the current price of a European hybrid put with twofold default risk is as follow

$$P(t)=C(t)+\delta(B(t,T)\rho-S(t))-(1-\delta)S(t)\exp[\int_t^T \mathcal{Z}(u)du-\alpha_1 U]$$
$$+B(t,T)(1-\delta)\{(K-\rho)\exp[-\int_t^T(\mathcal{W}(u)-0.5|\varsigma(u)|^2)du]+\rho\exp[-\int_t^T Y(u)du]\}$$
$$+B(t,T)(K-\rho)\delta\exp[-\int_t^T(\mathcal{X}(u)-0.5|\xi(u)|^2)du]$$

where

$$\mathcal{Z}(u)=\Theta(u;-\alpha_0,-\alpha_1,-\alpha_2,\Phi(u),1,1,0);$$

$$\mathcal{W}(u)=\Theta(u;\lambda_0+\alpha_0,\lambda_1+\alpha_1,\lambda_2+\alpha_2,0,0,0,1);$$

$$\mathcal{X}(u)=\Theta(u;\lambda_0,\lambda_1,\lambda_2,0,0,0,1);\ \varsigma(u)=\Delta(u;\lambda_1+\alpha_1,\lambda_2+\alpha_2).$$

Proposition 2 demonstrates the implied relation between the price of hybrid calls and puts. Via this relation, investors learn how to make the strategies for replicating the payoff of hybrid options in constructing his portfolio. Also, hybrid put-call parity has several special cases similar to Proposition 1. For example, choosing coefficients λ_0, λ_1, λ_2, α_0, α_1, and α_2 to be zero can degenerate it as a standard put-call parity

$$\widetilde{P}(t)=\widetilde{C}(t)-S(t)+K\,B(t,T)$$

where $\widetilde{C}(t)$ denotes the current price of a standard European call with the explicit form

$$\widetilde{C}(t)=S(t)\cdot \mathrm{N}(\widetilde{d}_1)-K\cdot B(t,T)\cdot \mathrm{N}(\widetilde{d}_2)$$

and the arguments to the standard normal cumulative distribution are

$$\widetilde{d}_1=\Psi(\widetilde{\beta}(u),0,0,0,0);\ \widetilde{d}_2=\widetilde{d}_1-\sqrt{\int_t^T 2\widetilde{\beta}(u)du};$$

$$\widetilde{\beta}(u)=\Xi(u;0,1,0,0,0,0,0,0,0,0,0).$$

Such an interesting connection allows us to verify the validity of Proposition 2.

2.5 Special Cases

Within the present framework, default intensity functions are formed as a linear combination of an intercept term and two macroeconomic factors, including the spot interest rates and the abnormal returns on market index. The coefficients of intensity function determine the level of spontaneous default intensity as well as its sensitivities to the two economic variables. The role these coefficients play can be conceptualized as "control valves" for default risk. If the whole valves are shut down (with letting $\lambda_0=\lambda_1=\lambda_2=\alpha_0=\alpha_1=\alpha_2=0$), the behavior of option would be senseless to default risk, in line with a standard option. The assumption that the option issuer and underlying stock both are default-free can be tractably released by choosing nonzero values for intensity coefficients. The combination of varied choices of the value of coefficients determines what types of special case our model is reduced to. For introducing them explicitly, several corollaries are provided as follows.

Corollary 1. Conditioned on not being default up to the current time (i.e., $\tau_S > t$), the explicit solution for a non-vulnerable call on a defaultable stock (letting $\alpha_0=\alpha_1=\alpha_2=0$) is as follows:

$$C'(t) = S(t)\exp[\int_t^T Z'(u)\,du + \lambda_1 U]\,\mathrm{N}(d_1') - K\,B(t,T)\,\mathrm{N}(d_2')$$

where

$$Z'(u) = \Theta(u;\lambda_0,\lambda_1,\lambda_2,\xi(u),1,1,0);$$

$$d_1' = \Psi(\beta'(u),\lambda_1,\lambda_1,\lambda_1,\lambda_2);\ d_2' = \Psi(\Gamma'(u),0,0,\lambda_1,\lambda_2);$$

$$\beta'(u) = \Xi(u;\xi(u),1,0,0,1,\lambda_2,\hat{\sigma}_S,\lambda_0,\lambda_1,\lambda_2,\xi(v));$$

$$\Gamma'(u) = \Xi(u;0,-1,0,0,0,0,b(v,T),\lambda_0,\lambda_1,\lambda_2,0).$$

Corollary 2. Conditioned on not being default up to the current time (i.e., $\tau_i > t$), the explicit solution for a vulnerable call (letting $\lambda_0=\lambda_1=\lambda_2=0$) is as follows:

$$\overline{C}(t)=(1-\delta)\{S(t)\exp[\int_t^T \overline{Z}(u)du+(-\alpha_1)U]\mathrm{N}(\overline{d}_1)-KB(t,T)\exp[-\int_t^T \overline{Y}(u)du]\mathrm{N}(\overline{d}_2)\}$$
$$+\delta[S(t)\mathrm{N}(\overline{d}_3)-KB(t,T)\mathrm{N}(\overline{d}_4)]$$

where

$$\overline{Z}(u)=\Theta(u;-\alpha_0,-\alpha_1,-\alpha_2,\Phi(u),1,1,0);$$

$$\overline{Y}(u)=-\Theta(u;-\alpha_0,-\alpha_1,-\alpha_2,\Phi(u),0,0,1);$$

$$\overline{d}_1=\Psi(\overline{\beta}(u),0,-\alpha_1,0,0);\ \overline{d}_2=\Psi(\overline{\Gamma}(u),0,0,0,0);$$

$$\overline{d}_3=\Psi(\overline{\theta}(u),0,0,0,0);\ \overline{d}_4=\Psi(\overline{\phi}(u),0,0,0,0);$$

$$\overline{\beta}(u)=\Xi(u;\Phi(u),1,0,0,1,-\alpha_2,\hat{\sigma}_S,0,0,0,\Phi(v));$$

$$\overline{\Gamma}(u)=\Xi(u;\Phi(u),-1,1,\alpha_2,0,0,b(v,T),0,0,0,\Phi(v));$$

$$\overline{\theta}(u)=\Xi(u;0,1,0,0,1,0,\hat{\sigma}_S,0,0,0,0);$$

$$\overline{\phi}(u)=\Xi(u;0,-1,0,0,0,0,b(v,T),0,0,0,0).$$

Given varied choices of intensity coefficients, the above corollaries present special cases of our generalized option pricing model. Notably, their implications are different. Corollary 1 provides the pricing formula for a non-vulnerable call that only accounts for reference risk. This is exclusively for special cases where option writers have “deep pockets”, and hence, never default. Corollary 2 puts a special focus on the vulnerable call pricing, which has motivated a long list of studies (Johnson & Stulz, 1987; Hull & White, 1995; Klein, 1996; Rich, 1996; Klein & Inglis, 1999, 2001; Hui et al., 2003). One typical case of option issuance belong to this corollary is the interest-rate option issued by parties with limited paying capacity. Another important function of the two corollaries is to achieve the separation between the net reference- default effect and counterparty-default effect

on option price. Such a separation helps understand how the tradeoff between these two types of default risk affects option pricing.

3. NUMERICAL ANALYSIS

This section numerically examines the properties of option price with respect to twofold default risk. At the beginning of analysis, for brevity, let the time to maturity be $T-t=1$; the 1-year forward rate be $f(t,T)=1.6\%$; the correlation coefficients be $-\rho_{rI}=-\rho_{rS}=\rho_{IS}=0.5$; the sensitivity of default intensity to economic variables be $\lambda_1=\alpha_1=-0.1, \lambda_2=\alpha_2=-0.001$; the spontaneous default intensity of option writer and underlying stock be $\lambda_0=\alpha_0=0.2$, the strike price be $K=\$100$; the current price of underlying stock be $S(t)=\$100$; the current level of market index be $I(t)=6000$; the permanent volatility of stock be $\sigma_S=40\%$; the volatility of spot interest rate and market index be $\sigma_r=1.57\%$ and $\sigma_I=20\%$; the recovery rate be $\delta=0.5$; the rate of rebate for reference default be $\rho/K=100\%$; and the parameter of extended Vasicek model be $a=0.0254$.[12][13]

3.1 Characters of Reference and Counterparty Risk for Option Value

Given the combination of the reasonable range of model's parameters, Table 2 reports the value of hybrid options, as well as percentage changes in the option value $(C(t)/\tilde{C}(t)-1)\times 100\%$ due to default risk. From numbers of percentage changes, it is observed that the impact of reference default risk on a hybrid option's price is entirely different from that of counterparty default risk. Specifically,

[12] The evidences from Duffee (1998) and Kao (2000) indicate that yield spreads on corporate bonds are observed to have negative relation with the Treasury yields and index return. In view of this fact, we choose "negative" sign in selecting the value of intensity coefficients $\lambda_1, \lambda_2, \alpha_1, \alpha_2$.

[13] The value we choose for the parameter a is taken from Jarrow & Yu (2001).

reference risk generates credit premium on option value, while counterparty risk generates credit discount. The former effect significantly dominates the latter effect.

Table 2 Hybrid Option Value and Percentage Changes in Option Value Due to Twofold Default Risk

α_0	λ_0	Price		Percentage Change	
		Call	Put	Call	Put
0.1	0.1	$21.4232	$19.9221	30.7967%	34.5700%
	0.2	$29.3831	$27.8819	79.3945%	88.3373%
0.2	0.1	$20.4498	$19.0166	24.8536%	28.4535%
	0.2	$28.0479	$26.6147	71.2425%	79.7776%

Table 2 reports hybrid-option value and percentage changes in the option value resulting from twofold default risk with varying choices of spontaneous default intensity. Percentage changes are calculated as the ratio of difference in the price between hybrid options and standard options to the price of standard options. Except for spontaneous default intensities, the values of model parameters are chosen at their baseline levels. Data source: Numerical results from this paper.

The influence of counterparty risk is not surprising, consistent with most of prior studies (Johnson & Stulz, 1987; Hull & White, 1995; Klein, 1996; Rich, 1996; Klein & Inglis, 1999, 2001; Hui et al., 2003). The result implicates that higher possibilities of the writer's default lead more likely the holder of option suffering a fraction loss of the option value even the underlying asset does not default. The intensity of issuer's default risk, hence, is negatively related to the hybrid option value.

The effect of reference risk on the option's value is our innovation that seems much more complicated than the case of counterparty risk. Within the present model, reference risk impacts the option price in two dimensions. On the one hand, reference risk accompanied with debt financing makes underlying stock's return

more volatile via financial leverage.[14] On the other hand, reference risk creates possibilities that the terminal intrinsic value of option will be fixed at times earlier than its expiration. The former intuitively causes positive impact on the option value; whereas the latter has asymmetric impact. Such an early-fixed effect increases the put's value but decreases the call value, since the occurrence of reference default heavily knocks the underlying stock price down to zero, and then freeze the transactions on underlying stocks. The tradeoff between leverage effect and early-fixed effect determines the net effect of reference risk on call price. Observably, in most circumstances, the former dominates the latter. Also, it is observable that the reference-default premium on a put is clearly larger than that on a call due to asymmetry in the early-fixed effect.

3.2 Co-movement between Reference Risk and Counterparty Risk

Next discuss the co-movement between reference and counterparty risk impact from the argument concerning option pricing. To this end, separating the net effects of counterparty risk and reference risk from option value is required. Using Corollaries 1 and 2 thus achieves such a separation. Then we calculate percentage changes in option value resulted from reference and counterparty risk respectively with varying choice of 1-year forward rate and market index level. Because these two economic factors jointly drive the evolution of default intensity in the model, our calculation can help understand the implication of co-movement between reference and counterparty risk for option pricing. Table 3 compiles the numbers of calculation.

[14] See Huffman (1983) and Maia (2010) for positive empirical relation between financial leverage and equity risk.

Table 3 Co-movement between Counterparty Risk and Reference Risk: A Perspective on Percentage Changes in Option Values

$f(t,T)$	$I(t)$	Counterparty Default Impact		Reference Default Impact	
		Call	Put	Call	Put
1%	3000	-8.7138%	-8.6790%	88.8997%	94.7154%
	6000	-8.6851%	-8.6504%	88.5047%	94.2944%
	9000	-8.6684%	-8.6336%	88.2738%	94.0485%
3%	3000	-8.6316%	-8.5968%	85.5136%	103.2277%
	6000	-8.6029%	-8.5680%	85.1323%	102.7674%
	9000	-8.5861%	-8.5512%	84.9094%	102.4984%
5%	3000	-8.5492%	-8.5143%	82.3096%	112.3150%
	6000	-8.5205%	-8.4855%	81.9412%	111.8123%
	9000	-8.5037%	-8.4687%	81.7259%	111.5185%

Table 3 presents the impact of co-movement between counterparty risk and reference risk on hybrid option pricing given varied choices of level of macroeconomic variables. Percentage changes in option value due to counterparty risk are calculated as the ratio of difference in the price between a vulnerable option (Corollary 2) and a standard option to the standard-option price. Percentage changes in option value due to reference risk are calculated as the ratio of difference in the price between a non-vulnerable defaultable-stock option (Corollary 1) and a standard option to the standard-option price. The values of model's parameters are chosen at their baseline levels. Data source: Numerical results from this paper.

First consider the case of market index. Observe from table that, given a rising index level, the impacts of counterparty default and reference default on option price both decrease slightly, suggesting a positive co-movement between the two types of default. More specifically, when the index level varies from 3000 to 9000

with fixing forward rate at 1%, counterparty default impact on call (put) reduces from -8.7138% (-8.679%) to -8.668% (-8.633%), while reference default impact reduces to 88.2738% (94.0485%) from 88.8997% (94.715%). Such a result echoes with empirical evidences of credit contagion or the clustering of default, implicating that individuals in the same economy often default simultaneously (Hackbarth et al., 2006).

The consistency in the co-movement between two types of default risk, however, does not hold for the case of forward rate. An exception appears when analyzing the reference risk impact on a put price. If the forward rate climbs from 1% to 5% given the index level of 9000, percentage changes in the put price due to reference risk rise from 94.0485% to 111.5185%. The reason for this exception lies in the fact that the channels to transmit the effect of changes in the forward rate on option price are not limited to default intensity. Another channel to deliver such effect is pricing numeraire. More exactly, the rise in forward rate lowers the current value of default-free zero- coupon bond (i.e., pricing numeraire under forward martingale measure), and thus, increases option price, and vice versa. Unfortunately, it is unavailable for our model to achieve the explicit separation between these two types of forward rate-driven change in the net counterparty/reference risk impact on option price. If forward-rate impacts on the percentage changes in option price due to counterparty/reference risk delivered through default intensity were filtered via model, the consistency in the co-movement between twofold default risks could be held.

3.3 Characters of Reference and Counterparty Risk for Option Hedging

Now turn attention to option hedge against reference/counterparty risk. Hedging ratios of options (delta ratios) are derived as the partial differentiation of its value with respect to the underlying stock price. These ratios help measure how much underlying stock one should long or short to offset the stock-price risk on option issuance. Given the same combination of model parameters, Table 4 shows the deltas of hybrid options and percentage changes in deltas because of twofold default risk.

Table 4 Deltas of Hybrid Options and Percentage Changes in Deltas Due to Twofold Default Risk

α_0	λ_0	Deltas		Percentage Changes	
		Call	Put	Call	Put
0.1	0.1	0.7109	-0.2462	19.6491%	-39.3490%
	0.2	0.8832	-0.0738	48.6609%	-81.8162%
0.2	0.1	0.6786	-0.2350	14.2122%	-42.1060%
	0.2	0.8431	-0.0704	41.9055%	-82.6433%

Table 4 reports hybrid-option deltas and percentage changes in the option deltas resulting from twofold default risk with varying choices of spontaneous default intensity. Percentage changes are calculated as the ratio of difference in the delta between hybrid options and standard options to the delta of standard options. Except for spontaneous default intensities, the values of model parameters are chosen at their baseline levels. Data source: Numerical results from this paper.

Interestingly, the pattern of numbers in Table 4 echoes with Table 2. As the spontaneous intensity of reference default rises up, we observe that put deltas display negative sensitivity; while call deltas display positive sensitivity. Such an asymmetric behavior of option deltas implicates that if the early-fixed (leverage) effect dominates leverage (early-fixed) effect, the involvement of reference risk into option issuance will discourage (encourage) issuers' hedging intention, because early-fixed (leverage) effect decreases (increases) the volatility of expected intrinsic value of options. Also, observe that the sign of delta ratios of puts and calls are opposite. A positive call delta economically means that, to achieve efficient hedge, writers must hold a corresponded long position on the underlying stock in constructing their portfolios, and vice versa.

The negative relation between counterparty risk and option delta is not surprising. The size of hedging is smaller given a higher intensity of counterparty default, in line with most of existing literature on option pricing subject to counterparty risk; such as Johnson & Stulz (1987), Hull & White (1995), Klein

(1996), Rich (1996), Klein & Inglis (1999, 2001), and Hui et al. (2003). This implies that instead of concerning for whether perfect hedge can be reached, option issuers being more likely to default will undertake under-hedging strategies.

4. CONCLUSION

This paper proposes a generalized model with the presence of the intersection of market, counterparty and reference risk for option pricing. Using martingale method, the explicit pricing formula and the put-call parity for hybrid options are derived. The availability of the proposed model is multiple, which allows for pricing the standard option, vulnerable options, and non-vulnerable option on defaultable securities. Such a model feature helps achieve the separation between the net effect of reference risk and counterparty risk on option price and hedge ratios. Our model still allows for studying how the interaction between counterparty and reference default affects the pricing of option from arguments concerning the intersection of market and credit risk.

Numerical results elucidate the significance of twofold default risks in pricing the option. Counterparty risk not surprisingly makes options depreciate. But the impact of reference risk on option value is relatively more complicated and manifested in two ways. On the one hand, it makes the return on the underlying stocks more volatile via financial leverage to increase option value, which is termed as leverage effect. On the other hand, it delivers possibility that the expected intrinsic value of option is fixed at dates earlier than maturity to increase the put's value but to decrease the call's value, which is termed as early-fixed effect. The tradeoff between these two effects (the sum of these two effects) determines the net impact of reference risk on call's (put's) value. In summary, we find counterparty risk accounts for credit discounts on option value; while reference risk accounts for credit premiums. The latter significantly dominates the former.

Numerical results also contribute a well understanding of the implications of two -fold default risk for option hedging. As argued by earlier literature, counterparty risk lowers the size of hedging, meaning that an option issuer who is

more likely to default may put less effort in hedging activities. The involvement of reference risk into option issuance reduces hedging position constructed by put writers but enlarges that by call writers. The implication behind asymmetric reference-risk impacts on option hedging is: if the early-fixed (leverage) effect dominates leverage (early-fixed) effect, reference risk discourages (encourages) issuers' hedging intention, because early-fixed (leverage) effect shrinks (amplifies) the volatility of expected intrinsic value of options.

The proposed model still generates a positive co-movement between reference default risk and counterparty risk. Specifically, given a worse economic state (measured as market index level), we show that both these two types of default risk have stronger impact on option price. Such an outcome echoes with empirical implications of credit contagion as well as the clustering of default that individuals in the common market may default simultaneously when economy is in a recession.

Much works remains to be done. For example, measuring how big the premiums or discounts on option price are explained by credit risk and assessing the empirical performance of our model are two important topics for future research. The proposed model can be applied to several dimensions such as the valuation of vulnerable exotic options or decision-making on dynamic hedging. More difficult extensions include the pricing of credit derivatives and the combination with credit migration. We expect that more following studies and applications will be devoted pertaining to this subject.

References

Altman, E. I., 1983, **Corporate Financial Distress**, 1st, New York: John-Wiley.

Baldi, P., Caramellino, L., and Gabriella-Lovino, M., 1999, "Pricing General Barrier Options: A Numerical Approach Using Sharp Large Deviations," **Mathematical Finance**, Vol. 9, No. 1, 293-321.

Barone, E., Barone-Adesi, G., and Antonio, C., 1998, "Pricing Bonds and Bond Options with Default Risk," **European Financial Management**, Vol. 4, No. 2, 231-282.

Campbell, J. Y. and Taksler, G. B., 2003, "Equity Volatility and Corporate Bond Yields," **Journal of Finance**, Vol. 58, No. 6, 2321-2349.

Cetin, U., Jarrow, R., Protter, P., and Warachka, M., 2006, "Pricing Options in an Extended Black-Scholes Economy with Illiquidity: Theory and Empirical Evidence," **The Review of Financial Studies**, Vol. 19, No. 2, 493-529.

Chen, H., 2010, "Macroeconomic Conditions and the Puzzles of Credit Spreads and Capital Structure," **Journal of Finance**, Vol. 65, No. 6, 2171-2212.

Chen, H., Miao, J., and Wang, N., 2010, "Entrepreneurial Finance and Nondiver-sifiable Risk," **Review of Financial Studies**, Vol. 23, No. 12, 4348-4388.

Duffee, G., 1998, "The Relation between Treasury Yields and Corporate Bond Yield Spreads," **Journal of Finance**, Vol. 53, No. 6, 2225-2241.

Geman, H. and Yor, M., 1996, "Pricing and Hedging Double-Barrier Options: A Probabilistic Approach," **Mathematical Finance**, Vol. 6, No. 4, 365-378.

Guillaume, T., 2003, "Window Double Barrier Option," **Review of Derivatives Research**, Vol. 6, No. 1, 47-75.

Hackbarth, D., Miao, J., and Morellec, E., 2006, "Capital Structure, Credit Risk, and Macro- economic Conditions," **Journal of Financial Economics**, Vol. 82, No. 3, 519-550.

Heath, D., Jarrow, R., and Morton, A., 1992, "Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation," **Econometrica**, Vol. 60, No. 1, 77-105.

Heston, S., 1993, "A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options," **The Review of Financial Studies**, Vol. 6, No. 2, 327-343.

Huang, J. Z. and Kong, W., 2003, "Explaining Credit Spread Changes: New Evidence from Option-Adjusted Bond Indexes," **The Journal of Derivatives**, Vol. 11, No. 1, 30-44.

Huffman, L., 1983, "Operating Leverage, Financial Leverage, and Equity Risk," **Journal of Banking and Finance**, Vol. 7, No. 2, 197-212.

Hui, C. H., Lo, C. F., and Lee, H. C., 2003, "Pricing Vulnerable Black-Scholes Option with Dynamic Default Barriers," **The Journal of Derivatives**, Vol. 10, No. 2, 62-69.

Hull, J. and White, A., 1995, "The Impact of Default Risk on the Price of Options and Other Derivative Securities," **Journal of Banking and Finance**, Vol. 19, No. 1, 299-322.

Hung, M. W. and Liu, Y. H., 2005, "Pricing Vulnerable Options in Incomplete Markets," **Journal of Futures Markets**, Vol. 25, No. 1, 135-170.

Jarrow, R. and Yu, F., 2001, "Counterparty Risk and the Pricing of Defaultable Securities," **Journal of Finance**, Vol. 56, No. 5, 1765-1799.

Jarrow, R. and Turnbull, S., 1995, "Pricing Derivatives on Financial Securities Subject to Credit Risk," **Journal of Finance**, Vol. 50, No. 1, 53-85.

Jarrow, R. and Turnbull, S., 2000, "The Intersection of Market and Credit Risk," **Journal of Banking and Finance**, Vol. 24, No. 1-2, 271-299.

Johnson, H. and Stulz, R. M., 1987, "The Pricing of Option with Default Risk," **Journal of Finance**, Vol. 42, No. 2, 267-280.

Jorion, P. and Zhang, G., 2009, "Credit Contagion from Counterparty Risk," **Journal of Finance**, Vol. 64, No. 5, 2053-2087.

Kao, D. L., 2000, "Estimating and Pricing Credit Risk: An Overview," **Financial Analysts Journal**, Vol. 56, No. 4, 50-66.

Klein, P. and Inglis, M., 1999, "Valuation of European Options Subject to Financial Distress and Interest Rate Risk," **The Journal of Derivative**, Vol. 6, No. 1, 44-56.

Klein, P. and Inglis, M., 2001, "Pricing Vulnerable European Options when the Option's Payoff can Increase the Risk of Financial Distress," **Journal of Banking and Finance**, Vol. 25, No. 5, 993-1012.

Klein, P., 1996, "Pricing Black-Scholes Options with Correlated Credit Risk," **Journal of Banking and Finance**, Vol. 20, No. 7, 1211-1229.

Kraft, H. and Steffensen, M., 2007, "Bankruptcy, Counterparty Risk, and Contagion," **Review of Finance**, Vol. 13, No. 4, 209-252.

Kunitomo, N. and Ikeda, M., 1992, "Pricing Options with Curved Boundaries," **Mathematical Finance**, Vol. 2, No. 4, 275-298.

Kusuoka, S., 1999, "A Remark on Default Risk Model," **Advanced Mathematical Economics**, Vol. 1, No. 1, 69-82.

Lando, D., 1998, "On Cox Processes and Credit Risky Securities," **Review of Derivatives Research**, Vol. 3, No. 2-3, 99-120.

Leland, H., 1994, "Corporate Debt Value, Bond Covenants, and Optimal Capital Structure," **Journal of Finance**, Vol. 49, No. 4, 1213-1252.

Liao, S. L. and Wang, C. W., 2002, "Pricing Arithmetic Average Reset Option with Control Variates," **The Journal of Derivatives**, Vol. 1, Winter, 59-74.

Lo, C. F. and Hui, C. H., 2007, "Valuing Double Barrier Options with Time- Dependent Parameters by Fourier Series Expansion," **International Journal of Applied Mathematics**, Vol. 36, No. 1, 1-5.

Maia, M. V., 2010, **Cash-Flow Risks, Financial Leverage, and the Cross Section of Equity Returns**, Ph. D. Dissertation, University of Pennsylvania.

Merton, R., 1974, "On the Pricing of Corporate Debt: The Risk Structure of Interest Rates," **Journal of Finance**, Vol. 29, No. 2, 449-470.

Merton, R., 1976, "Option Pricing When Underlying Stock Returns are Discontinuous," **Journal of Financial Economics**, Vol. 3, No. 1-2, 125-144.

Musiela, M. and Rutkowski, M., 1997, **Martingale Method in Financial Modeling**, 1st, Berlin, DE: Springer.

Pelsser, A., 2000, "Pricing Double Barrier Options Using Laplace Transforms," **Finance and Stochastics**, Vol. 4, No. 1, 95-104.

Rich, D., 1996, "The Valuation and Behavior of Black-Scholes Options Subject to

Intertemporal Default Risk," **Review of Derivatives Research**, Vol. 1, No. 1, 25-59.

Scott, L., 1987, "Option Pricing When the Variance Changes Randomly: Theory, Estimation, and an Application," **The journal of Financial and Quantitative Analysis**, Vol. 22, No. 4, 419-438.

Wilson, T. C., 1997, "Portfolio credit risk," **Risk**, Vol. 10, No. 9, 111-117.

Wilson, T. C., 1997, "Portfolio credit risk," **Risk**, Vol. 10, No. 10, 56-61.

APPENDIX: PROOF OF PROPOSITION 1

Firstly utilize the HJM term structure model, and specify a forward rate process in which is consistent with the extended Vasicek spot rate model to derive the specific multi integrals:

$$r(u)=f(u,u)=f(t,u)+\int_t^u \sigma_f(v,u)\,dW^{\mathrm{T}}(v)\,,\ \sigma_f(v,u)=\exp[a(v-u)]\cdot\hat{\sigma}_r \qquad \text{(A.1)}$$

where $f(t,u)$ means for time-t forward rates, and σ_f plays the related volatility. By Ito's lemma, dynamics of the forward stock price $F_s(t,T)\equiv S(t)/B(t,T)$ and abnormal market index return $X(t)=\ln[I(t)/B(t)]$ are

$$dF_s(t,T)/F_s(t-,T)=[\hat{\sigma}_s-b(t,T)]\cdot dW^{\mathrm{T}}(t)-d[N(t\wedge\tau_s)-\Lambda_s^*(t\wedge\tau_s)] \qquad \text{(A.2)}$$

$$dX(t)=\hat{\sigma}_I\cdot[b(t,T)-\hat{\sigma}_I/2]dt+\hat{\sigma}_I\cdot dW^{\mathrm{T}}(t) \qquad \text{(A.3)}$$

With letting $d\nu(u)\equiv[\hat{\sigma}_s-b(u,T)]\cdot dW^{\mathrm{T}}(u)-d[N(u\wedge\tau_s)-\Lambda_s^*(u\wedge\tau_s)]$, rewriting (A.2) and (A.3) in term of integral representations has

$$F_S(t,T)=F_S(0,T)+\int_0^t F_S(u-,T)\,d\nu(u)$$

and

$$X(t)=X(0)+\int_0^t\{\hat{\sigma}_I\cdot[b(u,T)-\frac{1}{2}\hat{\sigma}_I]du+\hat{\sigma}_I\cdot dW^{\mathrm{T}}(u)\}\,.$$

Thus the unique solution $F_S(T,T)=\xi(\nu)_{(T)}$ to (A.2), referred as the Doleans' exponential of ν, equals

$$\xi(\nu)_{(T)}=\xi(\nu)_{(t)}\,1_{\{\tau_s>T\}}\exp\{\int_t^T[\hat{\sigma}_s-b(u,T)]\cdot dW^{\mathrm{T}}(u)-\frac{1}{2}\int_t^T\left|\hat{\sigma}_s-b(u,T)\right|^2du+\int_t^T d\Lambda_S^*(u\wedge\tau_S)\} \qquad \text{(A.4)}$$

Similarly, the unique solution to (A.3) is easily given as

$$X(t)=X(0)+\int_0^t\hat{\sigma}_I\cdot[b(u,T)-\frac{1}{2}\hat{\sigma}_I]du+\hat{\sigma}_I\cdot dW^{\mathrm{T}}(t)\,. \qquad \text{(A.5)}$$

Given equation (7) and $1_{\{t<\tau_i\le T\}}=1_{\{\tau_i>t\}}-1_{\{\tau_i>T\}}$, the value expression of a hybrid call can be decomposed as the following

$$\begin{aligned}C(t)=&B(t,T)(1-\delta)\{\mathrm{E}_{\mathrm{P_T}}[S(T)1_{\{S(T)\ge K\}\cap\{\tau_S>T\}\cap\{\tau_i>T\}}\,|\,G_t]-\mathrm{E}_{\mathrm{P_T}}(K1_{\{S(T)\ge K\}\cap\{\tau_S>T\}\cap\{\tau_i>T\}}\,|\,G_t)\}\\&+B(t,T)\delta\{\mathrm{E}_{\mathrm{P_T}}[S(T)1_{\{S(T)\ge K\}\cap\{\tau_S>T\}\cap\{\tau_i>t\}}\,|\,G_t]-\mathrm{E}_{\mathrm{P_T}}(K1_{\{S(T)\ge K\}\cap\{\tau_S>T\}\cap\{\tau_i>t\}}\,|\,G_t)\}\end{aligned}\tag{A.6}$$

For simplicity in organizing our derivation, take the term $\mathrm{E}_{\mathrm{P_T}}[S(T)1_{\{S(T)\ge K\}\cap\{\tau_S>T\}\cap\{\tau_i>T\}}\,|\,G_t]$ as an example and let this be notated by A1. Since $B(T,T)=1$, we know that $F_s(T,T)\equiv S(T)$. If so, derive A1 by replacing $S(T)$ with (A.4) has

$$\begin{aligned}\mathrm{A1}=\mathrm{E}_{\mathrm{P_T}}(&1_{\{S(T)\ge K\}\cap\{\tau_S>T\}\cap\{\tau_i>T\}}F_s(t,T)\exp\{\int_t^T[\hat{\sigma}_s-b(u,T)]\cdot dW^{\mathrm{T}}(u)-\frac{1}{2}\int_t^T\left|\hat{\sigma}_s-b(u,T)\right|^2du\\&+\int_t^T d\Lambda_S^*(u\wedge\tau_S)\}\,|\,G_t)\end{aligned}\tag{A.7}$$

Now assume that there is a measure $\widetilde{\mathrm{P}}$ on (Ω,F^W) equivalent to $\mathrm{P_T}$ with the involved Radon-Nikodym density process $\hbar_{\mathrm{T}}(u),\,t\le u\le T$ in which can be set by

$$\hbar_{\mathrm{T}}(u):=\left.\frac{d\,\widetilde{\mathrm{P}}}{d\,\mathrm{P_T}}\right|_{G_u},\ \mathrm{P_T}\text{-a.s.}$$

and satisfies

$$\hbar_{\mathrm{T}}(T)=\hbar_{\mathrm{T}}(t)+\int_{[t,T]}\hbar_{\mathrm{T}}(u-)[\hat{\sigma}_s-b(u,T)]\cdot dW^{\mathrm{T}}(u).$$

From Girsanov's theorem and by Ito formula, we yield

$$\hbar_{\mathrm{T}}(T)=\hbar_{\mathrm{T}}(t)\exp\{\int_t^T[\hat{\sigma}_s-b(u,T)]\cdot dW^{\mathrm{T}}(u)-\frac{1}{2}\int_t^T\left|\hat{\sigma}_s-b(u,T)\right|^2du\}\tag{A.8}$$

and for any $t\in[0,T]$,

$$\tilde{W}(t)=W^{\mathrm{T}}(t)-\int_0^t[\hat{\sigma}_s-b(u,T)]du\,.\tag{A.9}$$

Changing the underlying measure from $\mathrm{P_T}$ to $\widetilde{\mathrm{P}}$ based on (A.8), thus (A.7) becomes

$$\mathrm{A1}=\mathrm{E}_{\mathrm{P_T}}(1_{\{\mathrm{S}(T)\geq K\}\cap\{\tau_S>T\}\cap\{\tau_i>T\}}F_s(t,T)\exp\{\int_t^T d\Lambda_S^*(u\wedge\tau_S)\}\hbar_{\mathrm{T}}(T)/\hbar_{\mathrm{T}}(t)\,|\,G_t)$$
$$=\mathrm{E}_{\tilde{\mathrm{P}}}\{1_{\{\mathrm{S}(T)\geq K\}\cap\{\tau_S>T\}\cap\{\tau_i>T\}}F_s(t,T)\exp[\int_t^T d\Lambda_S^*(u)]\,|\,G_t\}$$

To apply the properties of Cox process (Lando, 1998), iterate the above condition expectations

$$\mathrm{A1}=F_s(t,T)\mathrm{E}_{\tilde{\mathrm{P}}}\{\mathrm{E}_{\tilde{\mathrm{P}}}\{1_{\{\mathrm{S}(T)\geq K\}\cap\{\tau_S>T\}\cap\{\tau_i>T\}}\exp[\int_t^T d\Lambda_S^*(u)]\,|\,F_T^W\vee H_T^S\vee H_t^i\}\,|\,G_t\}\quad(\mathrm{A.10})$$

Via Bayesian rule, since the joint information set $F_T^W\vee H_T^S\vee H_t^i\supseteq F_T^S\vee F_T^r\vee F_T^I\vee H_T^S$ and $\{\omega:S(T,\omega)\geq K\}\subseteq\{\omega:\tau_S(\omega)>T\}\subseteq\{\omega:\tau_S(\omega)>t\}$, (A.10) can be derived as the following

$$\mathrm{A1}=F_s(t,T)\mathrm{E}_{\tilde{\mathrm{P}}}\{1_{\{\mathrm{S}(T)\geq K\}\cap\{\tau_S>T\}}\exp[\int_t^T d\Lambda_S^*(u)]\mathrm{E}_{\tilde{\mathrm{P}}}\{1_{\{\tau_i>T\}}\,|\,F_T^W\vee H_T^S\vee H_t^i\}\,|\,G_t\}$$
$$=F_s(t,T)\mathrm{E}_{\tilde{\mathrm{P}}}\{1_{\{\mathrm{S}(T)\geq K\}\cap\{\tau_S>T\}\cap\{\tau_S>t\}\cap\{\tau_i>t\}}\mathrm{E}_{\tilde{\mathrm{P}}}(1_{\{\tau_i>T\}\cap\{\tau_i>t\}}\,|\,F_T^W\vee H_T^S)[\mathrm{E}_{\tilde{\mathrm{P}}}(1_{\{\tau_i>t\}}\,|\,F_T^W\vee H_T^S)]^{-1}$$
$$\times\exp[\int_t^T d\Lambda_S^*(u)]\,|\,G_t\}$$
$$=1_{\{\tau_i>t\}\cap\{\tau_S>t\}}F_s(t,T)\mathrm{E}_{\tilde{\mathrm{P}}}\{1_{\{\mathrm{S}(T)\geq K\}}\exp[\int_t^T d\Lambda_S^*(u)-\int_t^T d\Lambda_i^*(u)]\,|\,G_t\}$$

(A.11)

Given the solution of κ_u^i and κ_u^S, (A.11) has a further expression

$$\mathrm{A1}=1_{\{\tau_i>t\}\cap\{\tau_S>t\}}F_s(t,T)\mathrm{E}_{\tilde{\mathrm{P}}}(1_{\{\mathrm{S}(T)\geq K\}}\exp\{\int_t^T[\lambda_S^*(r(u),X(u))-\lambda_i^*(r(u),X(u))]du\}\,|\,G_t)$$
$$=1_{\{\tau_i>t\}\cap\{\tau_S>t\}}F_s(t,T)\mathrm{E}_{\tilde{\mathrm{P}}}(1_{\{\mathrm{S}(T)\geq K\}}\exp\{\int_t^T[(\lambda_0-\alpha_0)+(\lambda_1-\alpha_1)r(u)+(\lambda_2-\alpha_2)X(u)]\,du\}\,|\,G_t)$$

(A.12)

Applying (A.9) with (A.2) and (A.3), we yield

$$\int_t^T X(u)\,du=\int_t^T X(t)\,du+\int_t^T\int_t^u\hat{\sigma}_I\cdot\hat{\sigma}_S-|\hat{\sigma}_I|^2/2\,dv\,du+\int_t^T\int_t^u\hat{\sigma}_I\cdot d\tilde{W}(v)\,du$$

(A.13)

$$\int_t^T r(u)\,du = \int_t^T f(t,u)\,du - \int_t^T \hat{\sigma}_S \cdot b(u,T)\,du - \int_t^T b(u,T)\,d\tilde{W}(u) + U \tag{A.14}$$

where

$U = (R^2/4a)[4A(t,T) - A^2(t,T) - 3 + 2a(T-t)]$, $R = \hat{\sigma}_r/a$, $A(t,T) = \mathrm{e}^{a(t-T)}$.

Then, by Fubini's theorem, substituting (A.13) and (A.14) into (A.12) has

$$\begin{aligned} \text{A1} = 1_{\{\tau_i > t\}\cap\{\tau_S > t\}}\, \mathrm{E}_{\tilde{\mathrm{P}}}\{1_{\{\mathrm{S}(T)\geq K\}} \exp[\int_t^T \eta(u)\cdot d\tilde{W}(u) - \int_t^T |\eta(u)|^2/2\,du] \mid G_t\} \\ \times F_s(t,T)\exp[\int_t^T Z(u)\,du + (\lambda_1 - \alpha_1)U] \end{aligned} \tag{A.15}$$

where functions $Z(u)$ and $\eta(u)$ are

$$\begin{aligned} Z(u) \equiv (\lambda_0 - \alpha_0) + (\lambda_1 - \alpha_1)[f(t,u) - \hat{\sigma}_S \cdot b(u,T)] + (\lambda_2 - \alpha_2)\{X(t) + [\hat{\sigma}_I \cdot \hat{\sigma}_S - 0.5|\hat{\sigma}_I|^2](u-t)\} \\ + 0.5|\eta(u)|^2; \end{aligned}$$

$$\eta(u) \equiv -(\lambda_1 - \alpha_1)\,b(u,T) + (\lambda_2 - \alpha_2)\cdot\hat{\sigma}_I \cdot (T-u).$$

Again using the technique of measure change, consider another Radon-Nikodym derivative

$$\hat{\hbar}(u) := \left.\frac{d\hat{\mathrm{P}}}{d\tilde{\mathrm{P}}}\right|_{G_u}, \tilde{\mathrm{P}}\text{-a.s.}$$

with the integral representation

$$\hat{\hbar}(T) = \hat{\hbar}(t)\exp[\int_t^T \eta(u)\cdot d\tilde{W}(u) - \frac{1}{2}\int_t^T |\eta(u)|^2\,du] \tag{A.16}$$

Thus, from Girsanov's theorem

$$\hat{W}(t) = \tilde{W}(t) - \int_0^t \eta(u)\,du, \forall t \in [0,T]. \tag{A.17}$$

To change the measure from $\tilde{\mathrm{P}}$ to $\hat{\mathrm{P}}$, integrating (A.16) with (A.15) yields

$$\begin{aligned}\mathrm{A1} &= 1_{\{\tau_i>t\}\cap\{\tau_S>t\}} F_s(t,T)\exp[\int_t^T Z(u)\,du+(\lambda_1-\alpha_1)U]\mathrm{E}_{\hat{\mathrm{P}}}(1_{\{\mathrm{S}(T)\geq K\}}\,|\,G_t)\\ &= 1_{\{\tau_i>t\}\cap\{\tau_S>t\}} F_s(t,T)\exp[\int_t^T Z(u)\,du+(\lambda_1-\alpha_1)U]\hat{\mathrm{P}}(\mathrm{S}(T)\geq K\,|\,G_t)\end{aligned}$$

Similarly, applying (A.1)-(A.3) with (A.9) and (A.17) can have the dynamics of stochastic processes under $\hat{\mathrm{P}}$

$$\begin{aligned}F_s(u,T)=F_s(t,T)+\int_t^u F_s(v-,T)\,\{&[\hat{\sigma}_s-b(v,T)+\eta(v)]\cdot[\hat{\sigma}_s-b(v,T)]dv+[\hat{\sigma}_s-b(v,T)]\cdot d\hat{W}(v)\\ &-d[N(v\wedge\tau_s)-\Lambda_s^*(v\wedge\tau_s)]\}\end{aligned}$$

$$X(u)=X(t)+\int_t^u \hat{\sigma}_I\cdot[\eta(v)+\hat{\sigma}_s-\hat{\sigma}_I/2]\,dv+\int_t^u \hat{\sigma}_I\cdot d\hat{W}(v)$$

$$r(u)=f(u,u)=f(t,u)+\int_t^u \sigma_f(v,u)\cdot[\hat{\sigma}_s-b(v,T)+\eta(v)]dv+\int_t^u \sigma_f(v,u)\,d\hat{W}(v)$$

which gives that

$$\begin{aligned}F_s(T,T)=F_s(t,T)\,1_{\{\tau_s>T\}}\exp\{&\int_t^T[\hat{\sigma}_s-b(u,T)]\cdot d\hat{W}(u)+\frac{1}{2}\int_t^T\left|\hat{\sigma}_s-b(u,T)\right|^2du\\ &+\int_t^T[\hat{\sigma}_s-b(u,T)]\cdot\eta(u)\,du+\int_t^T d\Lambda_S^*(u\wedge\tau_S)\}\end{aligned}$$

(A.18)

$$\int_t^T X(u)\,du=\int_t^T X(t)\,du+\int_t^T\int_t^u \hat{\sigma}_I\cdot[\eta(v)+\hat{\sigma}_s-\hat{\sigma}_I/2]\,dv\,du+\int_t^T\int_t^u \hat{\sigma}_I\cdot d\hat{W}(v)\,du$$

(A.19)

$$\begin{aligned}\int_t^T r(u)\,du=\int_t^T f(t,u)\,du-\int_t^T \hat{\sigma}_S\cdot b(u,T)\,du+U+|R|^2(\lambda_1-\alpha_1)\,V\\ -R\cdot\hat{\sigma}_I(\lambda_2-\alpha_2)\int_t^T(T-u)(A(u,T)-1)\,du-\int_t^T b(u,T)\cdot d\hat{W}(u)\end{aligned}$$

(A.20)

Substitute (A.18)-(A.20) into $\mathrm{A1}$ for proceeding the derivation as

$$\begin{aligned}\mathrm{A1} = 1_{\{\tau_i>t\}\cap\{\tau_S>t\}}\,\hat{\mathrm{P}}\big(F_s(t,T)\exp\{\int_t^T[\hat{\sigma}_s - b(u,T)]\cdot\eta(u)\,du + \frac{1}{2}\int_t^T\left|\hat{\sigma}_s - b(u,T)\right|^2 du + \lambda_0(T-t)\\ + \lambda_1\int_t^T r(u)\,du + \lambda_2\int_t^T X(u)\,du + \int_t^T[\hat{\sigma}_s - b(u,T)]\cdot d\hat{W}(u)\} \geq K \,\big|\, G_t\big)\\ \times F_s(t,T)\exp[\int_t^T Z(u)\,du + (\lambda_1-\alpha_1)U]\\ = 1_{\{\tau_i>t\}\cap\{\tau_S>t\}}F_s(t,T)\exp[\int_t^T Z(u)\,du + (\lambda_1-\alpha_1)\,U]\,\mathrm{N}(d_1)\end{aligned}$$

(A.21)

where the argument to standard normal cumulative distribution is

$$\begin{aligned}d_1 = [\int_t^T\left|\hat{\sigma}_s - (1+\lambda_1)\,b(u,T) + \lambda_2(T-u)\hat{\sigma}_I\right|^2 du]^{-1/2}\{\int_t^T\beta(u)\,du + \lambda_1[U + (\lambda_1-\alpha_1)\left|R\right|^2 V]\\ + \ln[F_S(t,T)/K]\}\end{aligned}$$

and the function

$$\begin{aligned}\beta(u) = \eta(u)\cdot[\hat{\sigma}_S - b(u,T)] + \left|\hat{\sigma}_S - b(u,T)\right|^2\big/2 + \lambda_0 + \lambda_1\{f(t,u) - \hat{\sigma}_S\cdot b(u,T)\\ -R\cdot\hat{\sigma}_I(\lambda_2-\alpha_2)(T-u)[A(u,T)-1]\} + \lambda_2[X(t) + (u-t)(\hat{\sigma}_S\cdot\hat{\sigma}_I - \left|\hat{\sigma}_I\right|^2\big/2) + \int_t^u\eta(v)\cdot\hat{\sigma}_I\,dv].\end{aligned}$$

Thus (A.21) gives the explicit solution for $\mathrm{A1}$. Through the same procedure, other conditional expectations in (A.6) can be solved as follows:

$$\mathrm{E}_{\mathrm{P_T}}(K1_{\{\mathrm{S}(T)\geq K\}\cap\{\tau_S>T\}\cap\{\tau_i>T\}}\,|\,G_t) = K\exp[-\int_t^T Y(u)\,du]\,\mathrm{N}(d_2);$$

$$\mathrm{E}_{\mathrm{P_T}}[S(T)1_{\{\mathrm{S}(T)\geq K\}\cap\{\tau_S>T\}\cap\{\tau_i>t\}}\,|\,G_t] = F_S(t,T)\exp[\int_t^T X(u)\,du + \lambda_1 U]\,\mathrm{N}(d_3);$$

$$\mathrm{E}_{\mathrm{P_T}}(K1_{\{\mathrm{S}(T)\geq K\}\cap\{\tau_S>T\}\cap\{\tau_i>t\}}\,|\,G_t) = K\,\mathrm{N}(d_4);$$

And this completes the proof.

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